

## On the non-renormalization of the AdS radius

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JHEP09(2009)056

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## On the non-renormalization of the AdS radius

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ABSTRACT: We show that the relation between the 't Hooft coupling and the radius of AdS is not renormalized at one-loop in the sigma model perturbation theory. We prove this by computing the quantum effective action for the superstring on  $AdS_5 \times S^5$  and showing that it does not receive any finite  $\alpha'$  corrections. We also show that the central charge of the interacting worldsheet conformal field theory vanishes at one-loop.

KEYWORDS: AdS-CFT Correspondence, Sigma Models

ARXIV EPRINT: [0906.4572](https://arxiv.org/abs/0906.4572)

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**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	The effective action in bosonic and RNS string	3
<b>2</b>	<b>Pure spinor superstring in the <math>AdS_5 \times S^5</math> background</b>	<b>5</b>
<b>3</b>	<b>Effective action</b>	<b>5</b>
3.1	Matter	6
3.2	Ghost	7
3.3	Effective action	7
<b>4</b>	<b>Central charge</b>	<b>8</b>
<b>A</b>	<b>Notations</b>	<b>10</b>
<b>B</b>	<b>Map to momentum space</b>	<b>10</b>

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**1 Introduction**

In this paper we consider string theory on an AdS background and discuss the quantum corrections to the target space radius, in the sigma model perturbation theory. The embedding of the string worldsheet into the target space is described by a sigma model on a (super)coset manifold. String propagation on the coset manifold  $G/H$  can be described by a gauged WZW models. In bosonic WZW models, the level of the current algebra gets shifted at one-loop from  $k$  to  $k + \frac{1}{2}c_G$ , where  $c_G$  is the quadratic Casimir of the group  $G$  [1]. In the sigma model interpretation of WZW theory, the level is related to the radius of the target space manifold, which is the inverse of the sigma model coupling constant. Therefore, the classical relation  $R^2/\alpha' = k$  gets modified at one-loop to  $R^2/\alpha' = k + \frac{1}{2}c_G$  and in the full quantum theory there is a minimal value for the radius of the manifold, set by the quadratic Casimir of the group. The situation is different for gauged WZW models with worldsheet supersymmetry [2]. In that case, the fermionic and bosonic determinants cancel out and the relation between the radius and the level is not renormalized. These kinds of sigma model describe bosonic or RNS string theory on backgrounds supported by NS-NS flux. What happens with Ramond-Ramond flux?

In this paper, we address this question in the case of superstring theory on  $AdS_5 \times S^5$ , which is described by a sigma model on a supercoset. The AdS radius is again equal to the inverse of the sigma model coupling constant and is related to the 't Hooft coupling  $\lambda$  of the dual  $\mathcal{N} = 4$  super Yang-Mills theory through the dictionary

$$R^2/\alpha' = f(\lambda), \quad f(\lambda) \sim_{\lambda \rightarrow \infty} \sqrt{\lambda} + C_1 + \mathcal{O}(1/\sqrt{\lambda}),$$

The leading term in the large 't Hooft coupling expansion corresponds to the classical supergravity dictionary, but in principle subleading terms are allowed and  $C_1$  would arise at one-loop in the sigma model perturbation theory. This would be the analogue of the finite shift by  $\frac{1}{2}c_G$  in the level of the current algebra in gauged WZW models. In this paper, we will show that

$$C_1 = 0 .$$

The fact that the classical  $AdS_5 \times S^5$  solution of type IIB superstring is not modified by higher order  $\alpha'$  corrections has been first discussed in the early days of AdS/CFT [3, 4]. The first correction to type IIB supergravity comes at  $\mathcal{O}(\alpha'^3)$  and it is the familiar  $R^4$  term. The only component of the curvature that enters the  $R^4$  term is proportional to the Weyl tensor, and since  $AdS_5 \times S^5$  is conformally flat, such leading correction vanishes. All the other terms related to  $R^4$  by supersymmetry also vanish in this background, as well as the corresponding higher order corrections to the dilaton equation of motion. Using superspace techniques, due to the 32 supersymmetries of this background, this result can be extended to prove that the solution is not renormalized at all orders in  $\alpha'$ .<sup>1</sup> More recently, S-duality arguments applied to the giant magnon dispersion relation (where the function  $f(\lambda)$  appears) have confirmed this result from the dual field theory side [6].

In order to study the renormalization of the radius, we need to compute the sigma model quantum effective action

$$S_{\text{eff}} = S_{\text{div}} + S_{\text{finite}} .$$

The divergent part of the effective action vanishes [7, 8], which implies that the sigma model is conformally invariant and the radius does not run (see also [9, 10]). However, one still needs to evaluate the finite part of the effective action, which may consist of local as well as non-local terms. The local terms can be reabsorbed or adjusted by local counter-terms to restore the classical symmetries. On the other hand, the presence of finite non-local contributions to the effective action could not be removed and would generate a non-zero  $C_1$ . Moreover, finite non-local terms in the effective action may produce gauge or BRST anomalies.

In this paper we will compute the finite part of the effective action at one-loop and show that all non-local contributions vanish. Due to the presence of the Ramond-Ramond flux, the worldsheet supersymmetric RNS description is not valid and we must use either the  $\kappa$ -symmetric Green-Schwarz-Metsaev-Tseytlin sigma model [11] or the BRST-symmetric pure spinor sigma model [12]. Since we would like to preserve covariance at all stages, we will consider the pure spinor approach. Because the covariant approach does not have worldsheet supersymmetry, we cannot borrow the RNS results, but we need to compute explicitly the one-loop effective action.

As a byproduct of our analysis, we will show that there are no gauge nor BRST anomalies in the sigma model, confirming by an explicit one-loop computation the all-loop algebraic arguments in [8]. The last step in checking that the sigma model is quantum mechanically consistent at one-loop is the determination of its central charge, namely the leading quartic

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<sup>1</sup>The non-renormalization is also confirmed by explicit computations of the OPE's of the currents [5].

pole in the OPE of two stress tensors. Using the background field method, we show that the central charge vanishes.<sup>2</sup>

Let us briefly comment on the case of the  $AdS_4 \times CP^3$  background. This is a solution of type IIA supergravity with only 24 supersymmetries, so the superspace arguments in [3, 4] do not hold and one might expect the solution to get corrected. Even if the background is realized as the supercoset  $Osp(4|6)/SO(1,3) \times U(3)$ , where  $Osp(4|6)$  has a vanishing dual Coxeter number just as  $PSU(2,2|4)$ , the full superstring sigma model is not described by a supercoset [14], unlike the  $AdS_5 \times S^5$  background. Hence, the methods we use in this paper may not be immediately generalized to that background. In [15], a correction to the function  $f(\lambda) = \sqrt{\lambda}(1 + C_1/\sqrt{\lambda} + C_2/\lambda + \dots)$  has been proposed, where  $C_1 = 0$  and  $C_2$  is a two loop numerical coefficient. It would be interesting to study this correction from the sigma model point of view.

In the rest of the introduction, we will review the computation of the effective action in the bosonic and RNS string. In section 2 and in the appendix we collect some notations about the superstring sigma model on  $AdS_5 \times S^5$ . In section 3 we compute the one-loop effective action using the background field method and discuss its properties. In section 4 we show that the central charge vanishes at one-loop.

### 1.1 The effective action in bosonic and RNS string

Let us review the computation of effective actions in the closed bosonic and RNS string. We will set the notations and show why the bosonic string renormalizes, while worldsheet supersymmetry protects the metric from  $\alpha'$  corrections at one-loop.

The bosonic string in a curved background is (we are assuming that  $B_{mn} = 0$ )

$$S_{\text{bos}} = \int d^2z [\partial x^m \bar{\partial} x^n G(x)_{mn}]. \tag{1.1}$$

In the covariant background field expansion we fix a classical solution of the worldsheet equations of motion  $x_0$  and expand around it in the quantum fluctuations  $X$ ,

$$S_{\text{bos}} = S_0 + \int d^2z \eta_{ab} [\nabla X^a \bar{\nabla} X^b + \dots], \tag{1.2}$$

where  $\dots$  are terms depending on the curvature,  $\nabla X^a = \partial X^a + A^{ab} X_b$ ,  $A^{ab} = \partial x_0^m \omega_m^{ab}$  and  $\omega_m^{ab}$  is the spin connection. When one uses the normal coordinate expansion within the background field method, local Lorentz invariance is used to fix the spin connection to zero. In this case the resulting effective action will not have this symmetry. Keeping local Lorentz invariance we have to check if the effective action is not anomalous under this symmetry.

The effective action in momentum space is

$$S_{\text{eff}} = \frac{1}{2} \int d^2k \left[ \frac{1}{2} A^{ab}(-k) A_{ab}(k) \frac{\bar{k}}{k} + \frac{1}{2} \bar{A}^{ab}(-k) \bar{A}_{ab}(k) \frac{k}{\bar{k}} - \frac{1}{2} A^{ab}(-k) \bar{A}_{ab}(k) \right] \tag{1.3}$$

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<sup>2</sup>The same method used in this paper can be applied to the  $AdS_p \times S^p$  pure spinor compactifications of [13], to obtain the same non-renormalization of the radius, due to the fact that the dual Coxeter numbers of the corresponding lower dimensional coset models vanish as well. For the non-critical  $AdS_{2p}$  backgrounds in [13], the dual Coxeter number does not vanish and the radius may get renormalized already at one loop.

The loop integrals are done using dimensional regularization adding a small mass  $m$  to the  $X^a$  fields in order to regularize IR divergencies. All the UV divergences cancel, and the dependence on the dimensional regularization mass scale  $\mu$  is an infrared effect, so we can identify the mass regulator  $m$  with  $\mu$ .<sup>3</sup> The gauge variation of the effective action vanishes, even for the non-local IR divergent terms. We see that there is a finite local counter-term responsible for the gauge invariance. This is just a redefinition of the metric

$$G(x_0)_{mn} \rightarrow \tilde{G}(x_0)_{mn} = G(x_0)_{mn} + \alpha' \frac{1}{4} \omega_m^{ab} \omega_{nab},$$

and the new metric now has a gauge transformation

$$\delta \tilde{G}(x_0)_{mn} = \alpha' \frac{1}{4} \partial_m \Lambda_{ab} \omega_n^{ab} + \alpha' \frac{1}{4} \partial_n \Lambda_{ab} \omega_m^{ab}.$$

The anomaly is trivial, this is the reason why we can fix the connection to be zero when using normal coordinates.

Let us see what happens in RNS string. Its action in a curved background is

$$S_{\text{RNS}} = \int d^2z \left[ \left( \partial x^m \bar{\partial} x^n + \frac{1}{2} \psi^m \bar{\partial} \psi^n + \frac{1}{2} \bar{\psi}^m \partial \bar{\psi}^n \right. \right. \tag{1.4} \\ \left. \left. + \frac{1}{2} \psi^m \Gamma_{op}^n(x) \bar{\partial} x^o \psi^p + \frac{1}{2} \bar{\psi}^m \Gamma_{op}^n(x) \partial x^o \bar{\psi}^p \right) G(x)_{mn} + \frac{1}{4} R_{mnop} \psi^m \psi^n \bar{\psi}^o \bar{\psi}^p \right].$$

Again, we fix a classical solution of the worldsheet equations of motion  $(x_0, \psi_0, \bar{\psi}_0)$  and expand around it in the quantum fluctuations  $(X, \Psi, \bar{\Psi})$ ,

$$S_{\text{RSN}} = S_0 + \int d^2z \left[ \eta_{ab} \nabla X^a \bar{\nabla} X^b + \eta_{ab} \frac{1}{2} \Psi^a \bar{\nabla} \Psi^b + \eta_{ab} \frac{1}{2} \bar{\Psi}^a \nabla \bar{\Psi}^b + \dots \right], \tag{1.5}$$

where ... are terms depending on the curvature. The effective action is just

$$-\frac{1}{2} \int d^2k [A^{ab}(-k) \bar{A}_{ab}(k)]$$

since non-local terms cancel due to worldsheet supersymmetry. We have to add a local counter-term to cancel the anomalous variation of this term, which removes the term above. Such term may appear naturally in other regularization scheme, see e.g. [17]. We see that in the case of RNS superstring, even without gauge fixing the connection, there are no finite local corrections in the effective action. In RNS the IR divergent terms are also present and are gauge invariant. These terms are related to the fact that  $X(z)$  is not a primary field in two dimensions. In the case case of Type I or Heterotic string we would have the usual local Lorentz anomaly that appears because we have only left moving fermions, which can be canceled by a variation of the  $B$  field. In coset models, like the superstring in  $AdS_5 \times S^5$  space, it is useful to keep the connection unfixed since this simplifies significantly the background field expansion.

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<sup>3</sup>We are ignoring IR divergent terms like  $\ln(\frac{|k|^2}{\mu^2})$ . These terms are an IR effect and are expected to vanish when the full perturbative series is summed [16].

## 2 Pure spinor superstring in the $AdS_5 \times S^5$ background

The  $AdS_5 \times S^5$  background can be described by the coset superspace  $PSU(2, 2|4)/SO(4, 1) \times SO(5)$  [11]. From the metric and structure constants listed in appendix A, we see that the super Lie algebra  $PSU(2, 2|4)$  admits a decomposition [18] under  $\mathbf{Z}_4$ ,  $\mathcal{H} = \sum \mathcal{H}_i$ ,  $i = 0$  to 3

$$\mathbf{T}_{\underline{a}} \in \mathcal{H}_2, \quad \mathbf{T}_{[ab]} \in \mathcal{H}_0, \quad \mathbf{T}_\alpha \in \mathcal{H}_1, \quad \mathbf{T}_{\hat{\alpha}} \in \mathcal{H}_3. \quad (2.1)$$

Using the supertrace notation and setting  $\alpha' = 1$ , we write the  $AdS_5 \times S^5$  pure spinor action [7, 19] as

$$S_0 = \frac{R^2}{2\pi} \int d^2z \text{Str} \left( \frac{1}{2} J_2 \bar{J}_2 + \frac{3}{4} J_3 \bar{J}_1 + \frac{1}{4} J_1 \bar{J}_3 + w \bar{\nabla} \lambda + \hat{w} \nabla \hat{\lambda} - N \hat{N} \right) \quad (2.2)$$

where

$$\begin{aligned} J_0 &= (g^{-1} \partial g)^{[ab]} \mathbf{T}_{[ab]}, & J_1 &= (g^{-1} \partial g)^\alpha \mathbf{T}_\alpha, \\ J_2 &= (g^{-1} \partial g)^m \mathbf{T}_m, & J_3 &= (g^{-1} \partial g)^{\hat{\alpha}} \mathbf{T}_{\hat{\alpha}}, \\ w &= w_\alpha \mathbf{T}_{\hat{\alpha}} \delta^{\alpha \hat{\alpha}}, & \lambda &= \lambda^\alpha \mathbf{T}_\alpha, & N &= -\{w, \lambda\}, \\ \bar{J}_0 &= (g^{-1} \bar{\partial} g)^{[ab]} \mathbf{T}_{[ab]}, & \bar{J}_1 &= (g^{-1} \bar{\partial} g)^\alpha \mathbf{T}_\alpha, \\ \bar{J}_2 &= (g^{-1} \bar{\partial} g)^m \mathbf{T}_m, & \bar{J}_3 &= (g^{-1} \bar{\partial} g)^{\hat{\alpha}} \mathbf{T}_{\hat{\alpha}}, \\ \hat{w} &= \hat{w}_{\hat{\alpha}} \mathbf{T}_\alpha \delta^{\alpha \hat{\alpha}}, & \hat{\lambda} &= \hat{\lambda}^{\hat{\alpha}} \mathbf{T}_{\hat{\alpha}}, & \hat{N} &= -\{\hat{w}, \hat{\lambda}\}, \\ \nabla &= \partial + [J_0, \quad ], & \bar{\nabla} &= \bar{\partial} + [\bar{J}_0, \quad ], \end{aligned} \quad (2.3)$$

$\delta_{\alpha\hat{\beta}} = (\gamma^{01234})_{\alpha\hat{\beta}}$ ,  $\mathbf{T}_A$  are the  $PSU(2, 2|4)$  Lie algebra generators. Note that

$$\{\mathbf{T}_\alpha, \mathbf{T}_\beta\} = \gamma_{\alpha\beta}^m \mathbf{T}_m, \quad \{\mathbf{T}_{\hat{\alpha}}, \mathbf{T}_{\hat{\beta}}\} = \gamma_{\hat{\alpha}\hat{\beta}}^m \mathbf{T}_m, \quad \{\mathbf{T}_\alpha, \mathbf{T}_{\hat{\beta}}\} = \left( \frac{1}{2} \gamma^{[ab]} \gamma^{01234} \right)_{\alpha\hat{\beta}} \mathbf{T}_{[ab]}. \quad (2.4)$$

Also note that  $\lambda$  and  $\hat{\lambda}$  are fermionic since  $(\mathbf{T}_\alpha, \mathbf{T}_{\hat{\alpha}})$  are fermionic and  $(\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}})$  are bosonic. The action of (2.2) is manifestly invariant under global  $PSU(2, 2|4)$  transformations which transform  $g(x, \theta, \hat{\theta})$  by left multiplication as  $\delta g = (\Sigma^A T_A) g$  and is also manifestly invariant under local  $SO(4, 1) \times SO(5)$  gauge transformations which transform  $g(x, \theta, \hat{\theta})$  by right multiplication as  $\delta_\Lambda g = g \Lambda$  and transform the pure spinors as

$$\delta_\Lambda \lambda = [\lambda, \Lambda], \quad \delta_\Lambda \hat{\lambda} = [\hat{\lambda}, \Lambda], \quad \delta_\Lambda w = [w, \Lambda], \quad \delta_\Lambda \hat{w} = [\hat{w}, \Lambda]$$

where  $\Lambda = \Lambda^{[ab]} T_{[ab]}$ .

## 3 Effective action

We can quantize the classical action (2.2) using the covariant background field method. This method was used in [7] to prove one-loop conformal invariance of (2.2). In this section we will compute the one-loop effective action.

### 3.1 Matter

A classical background field  $\tilde{g}$  is chosen and the quantum fluctuations are parameterized by  $X = X_1 + X_2 + X_3$ , with  $g = \tilde{g}e^X$ , in the gauge  $X_0 = 0$ . The quantum currents are

$$\begin{aligned} J &= g^{-1}\partial g = e^{-X}\tilde{J}e^X + e^{-X}\partial e^X, \\ \bar{J} &= g^{-1}\bar{\partial}g = e^{-X}\tilde{\bar{J}}e^X + e^{-X}\bar{\partial}e^X, \end{aligned} \quad (3.1)$$

where  $\tilde{J} = \tilde{g}^{-1}\partial\tilde{g}$ .

The OPE for the quantum fluctuations is

$$X^A(z)X^B(w) \rightarrow -\eta^{BA}\ln|z-w|^2. \quad (3.2)$$

Since we are going to do a one-loop computation, we expand (3.1) up to the second order in  $X$ ,

$$J|_i = \tilde{J}|_i + \frac{1}{R} \left( dX + [\tilde{J}, X] \right) |_i + \frac{1}{2R^2} \left[ dX + [\tilde{J}, X], X \right] |_i + \mathcal{O}(R^{-3}), \quad (3.3)$$

We separate the relevant terms in the action as follows. The kinetic terms are

$$S_{\text{kin}} = \int d^2z \text{Str} \left( \frac{1}{2} \nabla X_2 \bar{\nabla} X_2 + \frac{1}{4} \nabla X_1 \bar{\nabla} X_3 + \frac{3}{4} \nabla X_3 \bar{\nabla} X_1 \right), \quad (3.4)$$

where the covariant derivative  $\nabla X_i = \partial X_i + [J_0, X_i]$  depends on the background gauge current. We will put in  $S_I$  all the terms that contain either  $J_2$  or  $\bar{J}_2$  or both

$$\begin{aligned} S_I = \int d^2z \text{Str} \left( \frac{1}{2} J_2 [X_1, \bar{\nabla} X_1] + \frac{1}{2} \bar{J}_2 [X_3, \nabla X_3] \right. \\ \left. + \frac{1}{4} J_2 [[\bar{J}_2, X_1], X_3] - \frac{1}{4} J_2 [[\bar{J}_2, X_3], X_1] \right). \end{aligned} \quad (3.5)$$

To ease the notation, we will drop the  $\tilde{\phantom{x}}$  on top of the background currents. We will put into  $S_{\text{II}}$  all the terms that depend on  $J_1$  or  $\bar{J}_3$  or both

$$\begin{aligned} S_{\text{II}} = \int d^2z \text{Str} \left( \frac{1}{8} J_1 (3[X_1, \bar{\nabla} X_2] + 5[X_2, \bar{\nabla} X_1]) + \frac{1}{8} \bar{J}_3 (3[X_3, \nabla X_2] + 5[X_2, \nabla X_3]) \right. \\ \left. - \frac{1}{2} J_1 [[\bar{J}_3, X_2], X_2] + \frac{1}{4} J_1 [[\bar{J}_3, X_1], X_3] - \frac{1}{4} J_1 [[\bar{J}_3, X_3], X_1] \right). \end{aligned} \quad (3.6)$$

Finally, we collect in  $S_{\text{III}}$  the terms that depend on  $J_3$  or  $\bar{J}_1$  or both

$$\begin{aligned} S_{\text{III}} = \int d^2z \text{Str} \left( \frac{1}{8} \bar{J}_1 ([X_1, \nabla X_2] - [X_2, \nabla X_1]) + \frac{1}{8} J_3 ([X_3, \bar{\nabla} X_2] - [X_2, \bar{\nabla} X_3]) \right. \\ \left. + \frac{1}{2} \bar{J}_1 [[J_3, X_2], X_2] + \frac{3}{4} \bar{J}_1 [[J_3, X_1], X_3] + \frac{1}{4} \bar{J}_1 [[J_3, X_3], X_1] \right). \end{aligned} \quad (3.7)$$



### 3.2 Ghost

Let us consider the ghost part of the one loop effective action. We expand the left and right moving ghosts into upper case background fields and lower case fluctuations

$$(w, \lambda) \rightarrow (W + w, L + \lambda), \quad (\widehat{w}, \widehat{\lambda}) \rightarrow (\widehat{W} + \widehat{w}, \widehat{L} + \widehat{\lambda}). \quad (3.8)$$

The ghost Lorentz currents are expanded as

$$N \rightarrow N_{(0)} + \frac{1}{R}N_{(1)} + \frac{1}{R^2}N_{(2)}, \quad \widehat{N} \rightarrow \widehat{N}_{(0)} + \frac{1}{R}\widehat{N}_{(1)} + \frac{1}{R^2}\widehat{N}_{(2)}, \quad (3.9)$$

where  $(N_{(0)}, \widehat{N}_{(0)})$  denote the background currents while

$$\begin{aligned} N_{(1)} &= -\{W, \lambda\} - \{w, L\}, & N_{(2)} &= -\{w, \lambda\}, \\ \widehat{N}_{(1)} &= -\{\widehat{W}, \widehat{\lambda}\} - \{\widehat{w}, \widehat{L}\}, & \widehat{N}_{(2)} &= -\{\widehat{w}, \widehat{\lambda}\}. \end{aligned} \quad (3.10)$$

We expand the classical ghost action according to (3.10) and collect the terms quadratic in the fluctuations

$$\begin{aligned} S = \frac{1}{2} \int d^2z \text{Str} \left\{ N_{(0)} ([\overline{\nabla}X_3, X_1] + [\overline{\nabla}X_2, X_2] + [\overline{\nabla}X_1, X_3]) \right. \\ \left. + \widehat{N}_{(0)} ([\nabla X_3, X_1] + [\nabla X_2, X_2] + [\nabla X_1, X_3]) \right. \\ \left. - N_{(1)}\widehat{N}_{(1)} + N_{(2)}(\overline{J}_0 - \widehat{N}_{(0)}) + (J_0 - N_{(0)})\widehat{N}_{(2)} \right\}. \end{aligned} \quad (3.11)$$

### 3.3 Effective action

The computation of the effective action at one loop order proceeds as follows. There are two kind of terms that we need to compute, schematically

$$S_{\text{eff}} = \int d^2z \langle \mathcal{L}(z) \rangle - \frac{1}{2} \int d^2z \int d^2w \langle \mathcal{L}(z) \mathcal{L}(w) \rangle, \quad (3.12)$$

where the  $\langle \cdot \rangle$  denotes functional integration over the fluctuating fields. The first term  $\langle \mathcal{L}(z) \rangle$  corresponds to the normal ordering of the composite operators in the lagrangian: it is just given by the one loop self energy of the fluctuations at the same point, in operators with two external currents. These are the second lines in (3.5), (3.6) and (3.7). The second term  $\langle \mathcal{L}(z) \mathcal{L}(w) \rangle$  corresponds to the one loop fish diagram generated by the contraction of the operators with one external current, namely the first lines in (3.5), (3.6) and (3.7).

As an example, we explicitly evaluate the term in the effective action proportional to the operator  $\text{Str } J_1 \overline{J}_3$ . Using the OPE's for the quantum fluctuations (3.2) we find

$$\begin{aligned} S_{\text{eff}} = \int d^2z J_1^\alpha \overline{J}_3^{\widehat{\alpha}}(z) [-\ln(0)] \left( \frac{3}{4} f_{\widehat{\alpha}a}^\beta f_{\alpha\beta}^a + \frac{1}{4} f_{\widehat{\alpha}\beta}^{[ef]} f_{\alpha[ef]}^\beta \right) \\ + \int d^2z \int d^2w J_1^\alpha(z) \overline{J}_3^{\widehat{\alpha}}(w) f_{\widehat{\alpha}a}^\beta f_{\alpha\beta}^a \left( \frac{34}{64} \delta^{(2)}(z-w) \ln|z-w|^2 - \frac{30}{64} \frac{1}{|z-w|^2} \right), \end{aligned} \quad (3.13)$$

The last line in (3.6) contributes to the self-energy graphs in the first line, while the first line in (3.6) contributes through the OPE of  $\frac{3}{8}[X_1, \overline{\nabla}X_2] + \frac{5}{8}[X_2, \overline{\nabla}X_1]$  and  $\frac{3}{8}[X_3, \nabla X_2] +$

$\frac{5}{8}[X_2, \nabla X_3]$ , generating the term in the second line. Using the map to momentum space, listed in appendix B, this gives the following term in the effective action

$$S_{\text{eff}}^{(1)} = \int d^2z J_1^\alpha \bar{J}_3^{\hat{\alpha}} \left[ \frac{1}{\epsilon} \right] \left( \frac{3}{4} f_{\hat{\alpha}a}^\beta f_{\alpha\beta}^a + \frac{1}{4} f_{\hat{\alpha}\beta}^{[ef]} f_{\alpha[ef]}^\beta \right) - \int d^2z J_1^\alpha \bar{J}_3^{\hat{\alpha}} f_{\hat{\alpha}a}^\beta f_{\alpha\beta}^a \left[ 1 + \frac{1}{\epsilon} \right]. \quad (3.14)$$

The divergent part of the effective action cancels and we are left with the following finite piece<sup>4</sup>

$$S_{\text{eff}}(J_1 \bar{J}_3) = 10 \int d^2z \text{Str } J_1 \bar{J}_3, \quad (3.15)$$

which is local. By analogous computations we end up with the full effective action

$$S_{\text{eff}} = \int d^2z \text{Str} \left( a_1 J_2 \bar{J}_2 + a_2 J_1 \bar{J}_3 + \frac{1}{2} c_2(H) (J_0 \bar{J}_0 - N \bar{J}_0 - \hat{N} J_0) \right). \quad (3.16)$$

where  $a_1 = 8$ ,  $a_2 = 10$  and  $c_2(H) = 3$  is the quadratic Casimir of the gauge group  $H = \text{SO}(1,4) \times \text{SO}(5)$ . We did not include the IR singular terms proportional to  $\ln|p|^2/\mu^2$ , which are expected to vanish once the full perturbative series is included [16]. The expression (3.16) has the following properties:

- (i) It is local, hence it can be removed by adding a local counter-term  $S_c = -S_{\text{eff}}$  to the action, according to the prescription given in [8] to preserve gauge and BRST symmetries. As a result, we proved that there are no gauge nor BRST anomalies at one-loop.
- (ii) By explicit computation, we checked that operators of weight (2,0) and (0,2), e.g. of the kind  $\text{Str } J_i J_j$ ,  $\text{Str } N N$  or  $\text{Str } \hat{N} \hat{N}$ , are not generated at one-loop, due to remarkable cancellations in the diagrams caused by the vanishing of the dual Coxeter number of  $\text{PSU}(2,2|4)$ .

## 4 Central charge

The last step in checking that the worldsheet theory is consistent at one-loop is the computation of the central charge. The stress tensor for the action (2.2) is

$$T = -\text{Str} \left( \frac{1}{2} J_2 \bar{J}_2 + J_1 J_3 + w \nabla \lambda \right). \quad (4.1)$$

We want to compute the one loop correction to the central charge. This is the quartic pole in the OPE

$$\langle T(z) T(0) \rangle = \frac{c/2}{z^4} + \dots,$$

where  $\langle \cdot \rangle$  denotes functional integration. We expand  $T$  according to (3.1) and we compute the contractions of the fluctuations. The terms coming from the action do not contribute

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<sup>4</sup>We used the identity  $f_{\hat{\alpha}a}^\beta f_{\alpha\beta}^{\hat{\alpha}} = -f_{\hat{\alpha}\hat{\beta}}^a f_{\alpha a}^{\hat{\beta}}$  and the fact that the combination  $\frac{1}{2} f_{\hat{\alpha}a}^\beta f_{\alpha\beta}^a - \frac{1}{2} f_{\hat{\alpha}\beta}^{[ef]} f_{\alpha[ef]}^\beta = R_{\hat{\alpha}\alpha}(G) = 0$ , since  $G = \text{PSU}(2,2|4)$  has vanishing dual Coxeter number. Moreover we used the identity  $\gamma_{\hat{\alpha}\hat{\beta}}^m = \eta_{\alpha\hat{\alpha}} \eta_{\beta\hat{\beta}} (\gamma^m)^{\alpha\beta}$ .

to the central charge. We find a leading tree level contribution, proportional to  $1/R^4$ , where  $R$  is the radius and the action is normalized as  $S = \frac{R^2}{2\pi} \int \mathcal{L}$ . The one loop correction is proportional to  $1/R^6$ . To compute terms of order  $1/R^8$  we need to expand (3.1) up to  $\mathcal{O}(R^{-3})$ , so they will be neglected and we will stop at one loop. We find

$$\langle \frac{1}{2} \text{Str} J_2 J_2(z) \frac{1}{2} \text{Str} J_2 J_2(0) \rangle = \frac{1}{R^4} \frac{1}{z^4} \left( \frac{10}{2} - \frac{1}{2R^2} [1 + \ln|z - w|^2] \eta^{lm} f_{l\alpha}^{\hat{\delta}} f_{m\hat{\delta}}^{\alpha} \right), \quad (4.2)$$

where  $\eta^{lm} f_{l\alpha}^{\hat{\delta}} f_{m\hat{\delta}}^{\alpha} = \frac{1}{4} \text{Tr} \gamma^a \gamma_a = 40$ . The first term arises from the double contraction at tree level, while the second comes from the triple contraction at one loop. The second contribution is

$$\langle \text{Str} J_1 J_3(z) \text{Str} J_1 J_3(0) \rangle = -\frac{1}{R^4} \frac{1}{z^4} \left( \frac{32}{2} + \frac{1}{2R^2} [1 + \ln|z - w|^2] \eta^{lm} f_{l\alpha}^{\hat{\delta}} f_{m\hat{\delta}}^{\alpha} \right). \quad (4.3)$$

The mixed term is

$$-\langle \text{Str} J_2 J_2(z) \text{Str} J_1 J_3(0) \rangle = \frac{1}{R^6} \frac{1}{z^4} [1 + \ln|z - w|^2] \eta^{lm} f_{l\alpha}^{\hat{\delta}} f_{m\hat{\delta}}^{\alpha}. \quad (4.4)$$

By summing up (4.2), (4.3) and (4.4) we get the total contribution of the matter part. The one-loop correction cancels out exactly, leaving only the tree level part, which is the same as in flat space

$$\langle T_{\text{matter}}(z) T_{\text{matter}}(0) \rangle = -\frac{1}{R^4} \frac{22}{2z^4}. \quad (4.5)$$

Let us look at the ghost part. The tree level contribution involves a trace on the ghost spinor indices and is equal to the analogous flat space contraction. In the gauge  $X_0 = 0$  the ghost sector does not give any one-loop correction and it starts contributing only at two loops (the leading term in  $w[J_0, \lambda]$  with no external fields is  $\mathcal{O}(R^{-4})$ ), so we find

$$\langle T_{gh}(z) T_{gh}(0) \rangle = \frac{1}{R^4} \frac{22}{2z^4}, \quad (4.6)$$

and by adding (4.6) and (4.5) we proved that the total central charge vanishes at one loop.

Since the effective action does not receive any finite corrections at one loop, there is no correction to the stress tensor either.

## Acknowledgments

We would like to thank Daniel Nedel, Yaron Oz, Leonardo Rastelli and Matthias Staudacher for valuable discussions. We also thank Nathan Berkovits for collaboration during the initial stages of this work. The authors would like to thank the KITP at Santa Barbara for the kind hospitality, where this project has started. This research was supported in part by DARPA under Grant No. HR0011-09-1-0015 and by the National Science Foundation under Grant No. PHY05-51164.

## A Notations

The metric in the PSU(2, 2|4) is

$$\begin{aligned}
 \eta^{ab} &= \eta^{\underline{ba}}; & \eta_{ab} &= \eta_{\underline{ba}}; & (A.1) \\
 \eta^{\alpha\hat{\beta}} &= -\eta^{\hat{\beta}\alpha} = (\gamma_{01234})^{\alpha\hat{\beta}}; & \eta_{\alpha\hat{\beta}} &= -\eta_{\hat{\beta}\alpha} = (\gamma_{01234})_{\alpha\hat{\beta}}; \\
 \eta^{[ab][cd]} &= \frac{1}{2}\eta^{a[c}\eta^{d]b}; & \eta^{[a'b'][c'd']} &= -\frac{1}{2}\eta^{a'[c'}\eta^{d']b'}; \\
 \eta_{[ab][cd]} &= \frac{1}{2}\eta_{a[c}\eta_{d]b}; & \eta_{[a'b'][c'd']} &= -\frac{1}{2}\eta_{a'[c'}\eta_{d']b'}.
 \end{aligned}$$

The underlined vector index  $\underline{a} = 0, \dots, 9$  is ten-dimensional, while  $a = 0, \dots, 4$  and  $a' = 5, \dots, 9$  represent the  $AdS_5$  and the  $S^5$  directions respectively; the indices  $\alpha, \hat{\alpha} = 1, \dots, 16$  describe SO(1, 9) Weyl spinors of the same chirality. Capital letters are collective PSU(2, 2|4) indices,  $A = (\underline{a}, \alpha, \hat{\alpha}, [\underline{ab}])$ .

The metric satisfies  $\eta^{AB}\eta_{BC} = \delta_C^A$ . Denoting  $(\mathbf{T}_{\underline{a}}, \mathbf{T}_{[\underline{ab}]}, \mathbf{T}_{\alpha}, \mathbf{T}_{\hat{\alpha}})$  the generators of the algebra,  $\eta_{AB} = \langle \mathbf{T}_A, \mathbf{T}_B \rangle$ . The non-vanishing structure constants  $f_{AB}^C$  of the PSU(2, 2|4) algebra are

$$\begin{aligned}
 f_{\alpha\beta}^c &= \gamma_{\alpha\beta}^c, & f_{\hat{\alpha}\hat{\beta}}^c &= \gamma_{\hat{\alpha}\hat{\beta}}^c, & (A.2) \\
 f_{\alpha\hat{\beta}}^{[ef]} &= \frac{1}{2}(\gamma^{ef})_{\alpha}{}^{\gamma}\eta_{\gamma\hat{\beta}}, & f_{\alpha\hat{\beta}}^{[e'f']} &= -\frac{1}{2}(\gamma^{e'f'})_{\alpha}{}^{\gamma}\eta_{\gamma\hat{\beta}}, \\
 f_{\alpha\underline{c}}^{\hat{\beta}} &= -(\gamma_{\underline{c}})_{\alpha\beta}\eta^{\beta\hat{\beta}}, & f_{\hat{\alpha}\underline{c}}^{\beta} &= (\gamma_{\underline{c}})_{\hat{\alpha}\hat{\beta}}\eta^{\beta\hat{\beta}}, \\
 f_{cd}^{[ef]} &= \delta_c^{[e}\delta_d^{f]}, & f_{c'd'}^{[e'f']} &= -\delta_{c'}^{[e'}\delta_{d'}^{f']}, \\
 f_{[\underline{cd}][\underline{ef}]}^{[gh]} &= \frac{1}{2}(\eta_{\underline{ce}}\delta_{\underline{d}}^{[g}\delta_{\underline{f}]}^{h]} - \eta_{\underline{cf}}\delta_{\underline{d}}^{[g}\delta_{\underline{e}]}^{h]} + \eta_{\underline{df}}\delta_{\underline{c}}^{[g}\delta_{\underline{e}]}^{h]} - \eta_{\underline{de}}\delta_{\underline{c}}^{[g}\delta_{\underline{f}]}^{h]}) \\
 f_{[\underline{cd}]\underline{e}}^f &= \eta_{\underline{e}[\underline{c}}\delta_{\underline{d}]}^f, & f_{[\underline{cd}]\alpha}^{\beta} &= \frac{1}{2}(\gamma_{\underline{cd}})_{\alpha}{}^{\beta}, & f_{[\underline{cd}]\hat{\alpha}}^{\hat{\beta}} &= \frac{1}{2}(\gamma_{\underline{cd}})_{\hat{\alpha}}{}^{\hat{\beta}}.
 \end{aligned}$$

## B Map to momentum space

It is not clear upon inspection which terms in the effective action in coordinate space are finite or divergent. Also, we have the usual complications due to infrared divergencies. To clarify the interpretation, we will transform the above two point functions into loop integrals in the momentum space. We perform the loop integral using dimensional regularization adding a small mass  $m$  to the  $X^a$  fields in order to regularize IR divergencies. The dependence on the dimensional regularization mass scale  $\mu$  is an infrared effect, so we can identify the mass regulator  $m$  with  $\mu$ . In order to simplify the calculation, we build a dictionary between the above two point functions and the corresponding result of the

integration over the momenta [20]

$$\begin{aligned}
 \frac{1}{(z-w)^2} &\leftrightarrow -\frac{p}{\bar{p}}, \\
 \frac{1}{(\bar{z}-\bar{w})^2} &\leftrightarrow -\frac{\bar{p}}{p}, \\
 \frac{\ln|z-w|^2}{(z-w)^2} &\leftrightarrow -\frac{\bar{p}}{p} \left( 1 + 2\ln\left(\frac{|p|^2}{\mu^2}\right) \right), \\
 \frac{\ln|z-w|^2}{(\bar{z}-\bar{w})^2} &\leftrightarrow -\frac{p}{\bar{p}} \left( 1 + 2\ln\left(\frac{|p|^2}{\mu^2}\right) \right), \\
 \frac{1}{|z-w|^2} &\leftrightarrow 1 + \frac{1}{\epsilon} - 2\ln\left(\frac{|p|^2}{\mu^2}\right), \\
 -\ln|z-w|^2 \delta(z-w) &\leftrightarrow 1 + \frac{1}{\epsilon}, \\
 -\ln(0) &\leftrightarrow \frac{1}{\epsilon}.
 \end{aligned}
 \tag{B.1}$$

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